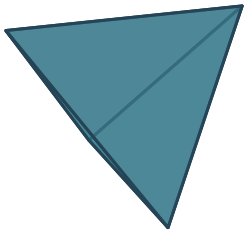
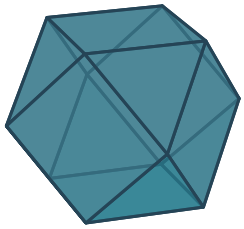
GESTALT SCIENCE / FORMATICS

# A Conceptual Tree of Symmetry, Form, Growth, Contraction, and Skewness

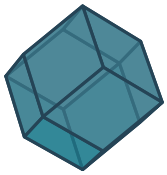
From polygons and polyhedra to finite informational geometries



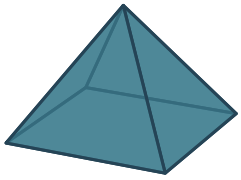
**Tetrahedron**
regular



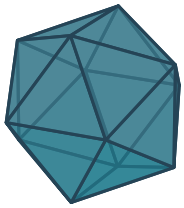
**Cuboctahedron**
uniform / Archimedean



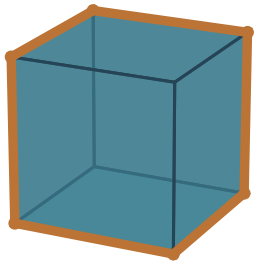
**Rhombic dodecahedron**
Catalan dual



**Square pyramid**
Johnson example



**Icosahedron**
Platonic seed



**Petrie cube**
skew Petrie circuit

Symmetry growth • symmetry contraction • skewness • transport
Riemannian • Weyl–Cayley • Ricci • Berry

Gestalt Science / Formatics Note | October 27, 2026

## Conceptual introduction

I think a particularly productive way to approach this is to treat polygons, polyhedra, polytopes, and their symmetry families not primarily as collections of geometric objects, but as **different routes through a landscape of finite informational forms**.

The central Formatic question becomes:

> *How does a finite structure acquire, preserve, transform, constrain, or lose symmetry as its relationships become increasingly complex?*

A cube, an icosahedron, an Archimedean solid, a Kepler–Poinsot star, a Petrie polygon, and a skew apeirohedron need not simply occupy separate boxes in a taxonomy. They can be understood as outcomes of different **processes of form generation**.

I propose organizing this exploration around four fundamental ideas:

1. **Symmetry growth:** Expansion of a form through new vertices, edges, facets, incidence relationships, or symmetry-preserving operations.
2. **Symmetry contraction:** Reduction, identification, quotienting, dualization, or selective elimination of geometric relationships.
3. **Skewness:** The inability—or deliberate refusal—of a structure's incidence and symmetry relationships to be represented within a specified planar, metric, or dimensional arrangement.
4. **Symmetry transport:** The propagation and transformation of local geometric relationships across a larger structure, potentially producing holonomy, chirality, and global constraints.

An important distinction: growth in geometric complexity does not necessarily mean growth in the order of the symmetry group. A form may acquire many more vertices and faces while retaining exactly the same symmetry group. Conversely, an apparently small combinatorial change may destroy most of its symmetry.

## 1. The two remarkable classification results

| **48** | **146** |
|---|---|
| **Regular skeletal polyhedra** | **Exceptional noble polyhedra** |
| Finite polyhedra, planar tessellations, and infinite skeletal forms. | Vertex-transitive and face-transitive forms, including many nonconvex examples. |

These counts are especially interesting because they are not merely different-sized collections of polyhedra. They arise from different definitions of symmetry.

The **48 regular skeletal polyhedra** in the Grünbaum–Dress classification consist of 18 finite examples, six planar examples, and 24 further infinite examples. This classification allows skeletal and skew constructions far beyond the traditional five Platonic solids.

The **146 exceptional noble polyhedra** come from Connor Hill's July 2026 computer-assisted classification. A noble polyhedron is both vertex-transitive and face-transitive, without necessarily being edge-transitive or fully regular. The 146 exceptional examples coexist with two infinite families, the disphenoids and stephanoids.

This suggests something important for Formatics:

> **There is no single hierarchy in which greater geometric complexity always produces either more symmetry or less symmetry. There are several partially independent dimensions of symmetry.**

That observation should become the foundation of our conceptual tree.

## 2. The symmetry tree: from simple incidence to complex form

Rather than organizing everything solely by number of dimensions, I would start with the primitive geometric relationships:

- **Vertex:** A localized distinguishable informational position.
- **Edge:** A relationship between two vertices.
- **Face:** A closed, ordered arrangement of edges.
- **Cell:** A bounded arrangement of faces.
- **Polytope:** A higher-dimensional incidence structure built by extending this hierarchy.

But the critical additional object is the *vertex figure*: what is happening in the immediate neighborhood of a vertex?

For instance, a regular icosahedron has five triangular faces meeting at every vertex. A cuboctahedron has triangles and squares alternating around every vertex. A Johnson solid can have several distinct kinds of vertex neighborhoods. That introduces complexity even when the ambient dimension remains three.

Strictly speaking, this should eventually become a directed graph or partially ordered lattice, not a literal tree. Families overlap: Platonic solids are simultaneously regular, uniform, and noble. Duality and Petrie transformations also produce lateral connections between branches.

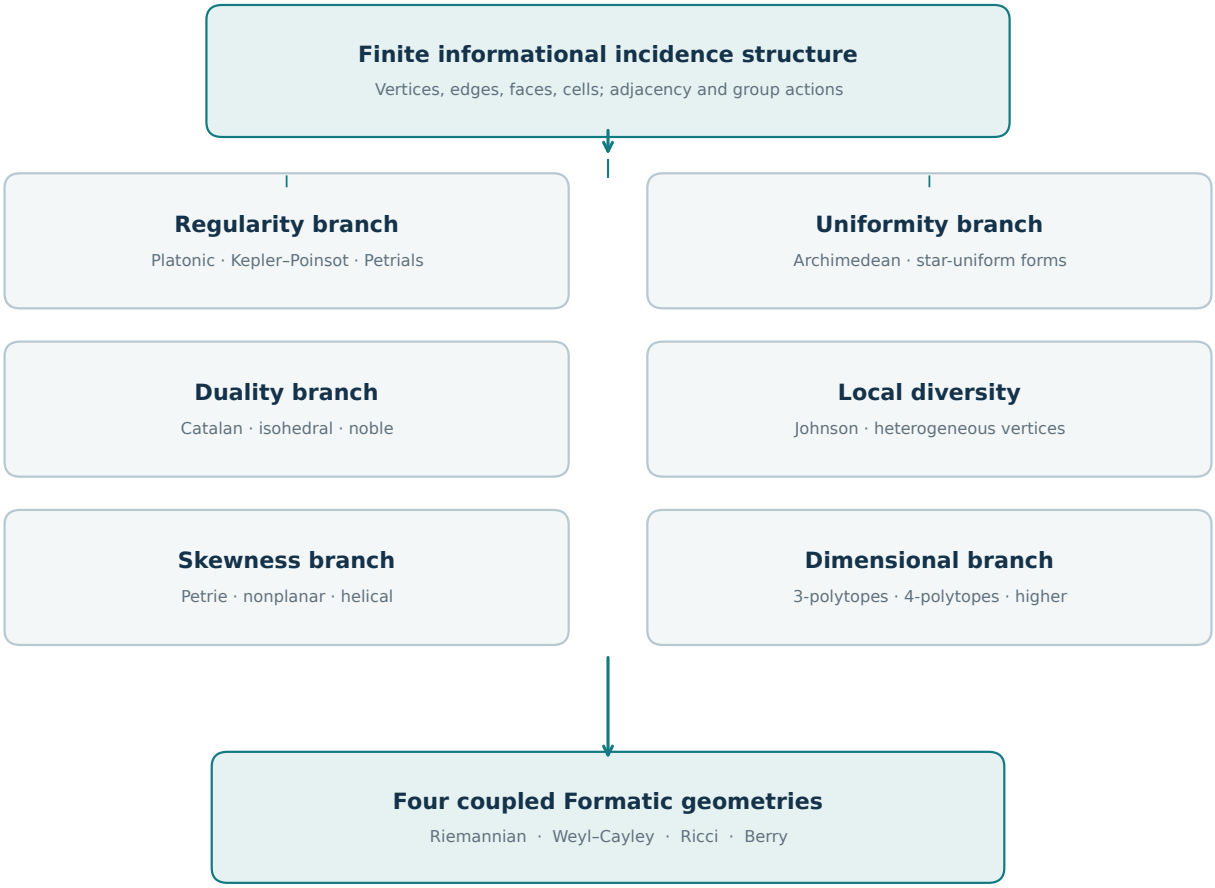



Figure 1: Proposed conceptual tree of Formatic geometry. The branches overlap and are not a strict partition.

## The principal families



Tetrahedron Cube Octahedron Dodecahedron Icosahedron

**Platonic solids — maximal classical regularity.**
Five convex solids, with one kind of regular face and equivalent vertices, edges, and flags. They form the central reference points for measuring symmetry preservation or symmetry breaking.

**Archimedean solids — uniform expansion.**
Thirteen convex solids with regular faces and equivalent vertices. Different face types may meet in the same ordered configuration at every vertex. Complexity expands while vertex symmetry survives.

**Catalan solids — face-centered duality.**
The thirteen duals of the Archimedean solids. Faces become equivalent while vertices can occupy different symmetry orbits. This transfers the organization of symmetry from vertices to faces.

**Johnson solids — local diversity.**
Ninety-two additional convex solids with regular polygon faces. Many have different vertex environments and much less global symmetry. The constituent faces remain regular while their assembly becomes heterogeneous.

**Kepler–Poinsot — nonconvex regularity.**
Four regular star polyhedra. Their intersecting and stellated geometry demonstrates that nonconvexity need not destroy regular symmetry.

**Grünbaum–Dress — skew and extended regularity.**
The broader 48-form classification includes skeletal configurations with skew polygons, infinite faces, and infinite arrangements. Traditional bounded solid geometry is no longer the defining requirement.

The familiar 75 nonprismatic uniform polyhedra and the 92 Johnson solids are established enumeration results.

The 48 regular skeletal polyhedra can be subdivided more precisely:

| Branch | Number |
|---|---|
| Finite regular forms, including Petrials | 18 |
| Planar tilings and their Petrials | 6 |
| Blended infinite regular forms | 12 |
| Pure infinite regular forms | 12 |
| **Total** | **48** |

The distinction depends on the chosen definition of a geometric polyhedron; the number 48 is not a universal count under every possible definition.

## 3. Four Formatic geometries operating on the same structure

Your four geometries are particularly useful because they suggest four different kinds of information about a form. I would give each a distinct mathematical job.

| Formatic geometry | Mathematical focus | Characteristic question |
|---|---|---|
| **Finite Riemannian** | Internal metric and incidence | What are the internal distances, angles, and neighborhood relationships? |
| **Finite Weyl–Cayley** | Conformal rescaling and symmetry generators | How can the form change scale or shape while preserving selected structural relationships? |
| **Finite Ricci** | Curvature distribution and normalization | How does local geometric irregularity constrain or redistribute globally? |
| **Finite Berry** | Connections, cyclic transport, and holonomy | What changes when information moves around the form and returns? |

These are proposed Formatic interpretations built from established discrete geometry, group theory, and geometric-phase mathematics. The combined four-geometry formalism is not presently a standard mathematical theory.

### A. Internal: finite Riemannian kernel

Start with a finite geometric complex

$$\mathcal{P} = (V, E, F, I, \ell),$$

where $I$ specifies incidence and $\ell$ assigns edge lengths.

This describes the intrinsic structure—what can be measured internally without depending on the larger space into which the form is embedded.

For a triangulated polyhedral surface, the curvature concentrated at an interior vertex is its angle defect:

$$K(v) = 2\pi - \sum_{f \ni v} \theta_f(v).$$

Consider a regular icosahedron. Five equilateral triangles meet at each vertex:

$$K(v) = 2\pi - 5\frac{\pi}{3} = \frac{\pi}{3}.$$

There are twelve vertices, so

$$\sum_v K(v) = 12\frac{\pi}{3} = 4\pi.$$

The local curvature contributions reconstruct a global invariant, consistent with discrete Gauss–Bonnet on the sphere.

In Formatics, I would interpret the angle defect as an **internal informational mismatch between local flatness and global closure**. This is measurable without assuming a smooth manifold.

## B. Conformal: finite Weyl–Cayley wrapper

Here I would distinguish two mathematical ingredients that are sometimes conflated.

A **Weyl-type geometry** concerns local metric rescaling and the transport of scale. A **Cayley graph** describes the action of a group through selected generators. These are not the same construction, but combining them could be fruitful.

Take a finite symmetry group $G$, with generators $S$:

$$\mathrm{Cay}(G, S).$$

The Cayley graph records which transformations connect group elements. It provides a discrete framework through which a geometric construction can propagate symmetry.

Separately, a discrete conformal transformation can be modeled, for suitable triangulated surfaces, by vertex scaling:

$$\ell'_{ij} = e^{(u_i+u_j)/2}\ell_{ij}.$$

This modifies the internal metric while retaining the labeled vertices and edges, subject to validity conditions such as triangle inequalities. Discrete conformal geometry and curvature flows provide rigorous precedents for such constructions.

For a strictly rational formulation, one could instead use finite positive rational scaling parameters and rational squared-length data. Such a model would be a proposed finite analog; it would require its own precise definition of conformal equivalence.

The Formatic interpretation becomes:

> **The Riemannian kernel describes a finite structure; the Weyl–Cayley wrapper describes permitted transformations of that structure.**

This is a useful role for your Weyl–Cauchy wrapper concept, provided we distinguish the metric, the symmetry group, and the causal or ordering relationships it introduces.

## C. Normalized: finite Ricci geometry

For a two-dimensional polyhedral surface, Ricci curvature can be related to Gaussian curvature. In discrete settings, several competing formulations exist.

A useful normalization of the angle defect is

$$\widehat{K}(v) = \frac{K(v)}{2\pi}.$$

For an icosahedral vertex:

$$\widehat{K}(v) = \frac{1}{6}.$$

This makes curvature an explicitly dimensionless local quantity.

Now consider a curvature redistribution process that seeks a prescribed target curvature $K^*(v)$. Schematically,

$$\frac{du_v}{dt} = K^*(v) - K(v).$$

With a suitable discrete conformal metric and flow convention, this resembles combinatorial Ricci flow.

The underlying idea would be a process of normalization: local curvatures evolve, but global topological and geometric compatibility requirements limit what can be achieved.

In Formatic language, Ricci normalization could function as **a finite process of redistributing local geometric incompatibilities under global structural constraints**.

Crucially, normalization does not always mean perfect uniformity. A Johnson solid, for example, may possess multiple vertex types that cannot all be made geometrically equivalent while preserving its complete original structure.

## D. Dynamic external: finite Berry regularization

This may be the most speculative—and interesting—of the four.

Berry geometry is fundamentally about how a state transforms when its parameters are transported around a closed path. Barry Simon's geometric interpretation explicitly connects Berry phase with holonomy.

For a finite Formatic complex, assign a transport element to every oriented edge:

$$U_{ij} \in H, \qquad U_{ji} = U_{ij}^{-1},$$

where $H$ is a finite transport group.

For a closed route

$$\gamma = (v_0, v_1, \dots, v_n = v_0),$$

define its holonomy by

$$\mathrm{Hol}(\gamma) = U_{01}U_{12}\cdots U_{n-1,0}.$$

The result measures how a transported informational state changes after completing a cycle.

This is a genuine finite, group-valued connection construction. Calling it *Berry regularization* would be a proposed Formatic extension; Berry phase itself usually involves continuous parameter families and quantum eigenstates.

What makes it relevant here is that a polyhedron provides many natural transport circuits: face boundaries, vertex links, Petrie polygons, and longer closed paths.

Thus, even when local vertex configurations appear identical, their global transport relationships can differ.

In your terminology, the external Berry layer could represent **the information accumulated by a form through relational journeys that cannot be reduced to its individual vertices, edges, or faces**.

## 4. A rational curvature framework for Formatics

There is a particularly appealing way to express finite polyhedral curvature entirely in rational numbers.

For a convex polyhedron assembled from regular polygonal faces, let the vertex configuration be

$$\nu(v) = (p_1, p_2, \dots, p_q),$$

where $p_i$ is the number of sides in the $i$-th incident face.

The normalized angle defect becomes

$$\boxed{\widehat{K}(v) = 1 - \sum_{i=1}^{q} \frac{p_i - 2}{2p_i}}$$

Every term is rational.

This gives us a finite informational curvature measure using $\mathbb{Q}$, without needing a continuous curvature field.

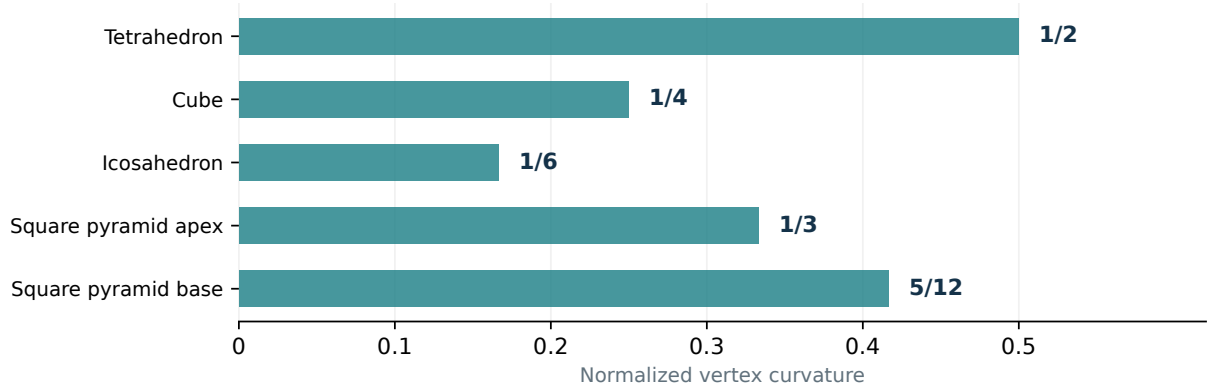



Figure 2: Exact rational vertex-curvature examples, with bars comparing the magnitudes.

A square pyramid, the first Johnson solid, makes an especially interesting example. It has two distinct vertex configurations: one apex of type $3.3.3.3$, and four base vertices of type $3.3.4$.

Its normalized curvature sums to

$$\frac{1}{3} + 4\left(\frac{5}{12}\right) = 2.$$

Thus the individual vertices are geometrically inequivalent, but the total curvature still satisfies

$$\sum_{v} \widehat{K}(v) = \chi = 2.$$

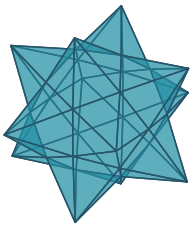
For a sphere-like polyhedral surface, this expresses discrete Gauss–Bonnet using rational arithmetic.

This is a promising starting point for your finite Ricci framework. More elaborate constructions could use discrete curvature flows while preserving a specified set of global constraints.

The limitation is that this simple formula assumes ordinary planar regular faces and their usual Euclidean angles. Skew faces, star polygons, and higher-dimensional cells need an appropriately extended curvature definition.

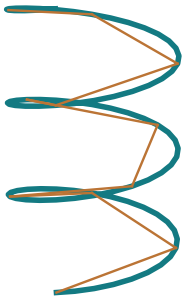
## 5. Skewness as an informational property

Your earlier proposition, **“the skew is information,”** becomes quite powerful in this taxonomy.



**Stellation-inspired nonconvex form**

schematic, not a named Kepler–Poinsot solid



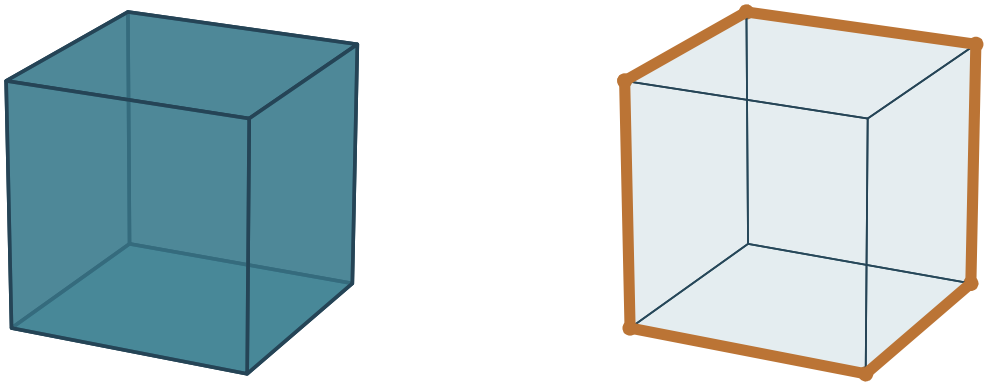
**Helical skew path**

nonplanar relational transport schematic

Figure 3: Original vector schematics illustrating nonconvex stellation and nonplanar helical extension. These sketches are not intended as classified regular polyhedra.

Petrie polygons offer a remarkably precise case. They are edge circuits in which every two consecutive edges, but no three consecutive edges, lie on one face.

Consider a cube:



Conventional cube Petrie hexagon (highlighted)

Figure 4: Same cubical edge structure, but distinct face-circuit organization. Orange traces a skew Petrie hexagon.

- The conventional cube has 8 vertices, 12 edges, and 6 square faces.
- Its Petrial retains the same 8 vertices and 12 edges, but substitutes 4 skew hexagonal faces.
- The Euler characteristic consequently changes from $8 - 12 + 6 = 2$ to $8 - 12 + 4 = 0$.

The point is not simply that the hexagons are skew. The same underlying wireframe supports different face-incidence structures, and those choices change the topology of the associated abstract surface.

This is especially relevant to Formatics: the geometry is not exhausted by the positions of its vertices or the lengths of its edges.

I would distinguish four varieties of skewness:

| **Variety** | Interpretation |
|---|---|
| **Geometric skew** | A path or polygon cannot lie in a common plane |
| **Incidence skew** | Different face associations are placed on the same underlying graph |
| **Dimensional skew** | A desired realization cannot be achieved within a specified ambient dimension or representation |
| **Transport skew** | Moving around closed routes produces nontrivial holonomy |

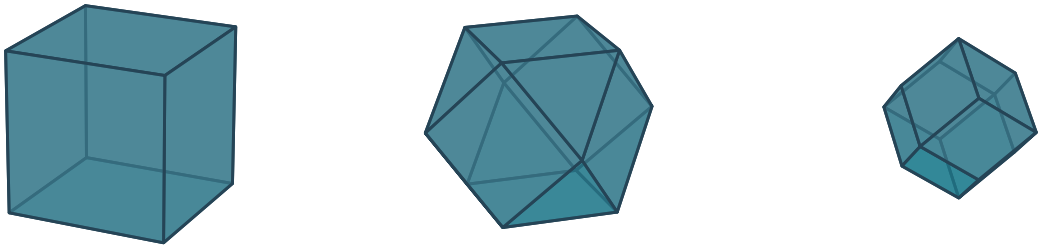
Only the first two directly describe familiar polyhedral skew constructions. The latter two would be new, explicitly defined Formatic extensions.

A form can therefore be perfectly symmetric under one group action while being skew under a particular geometric realization.

**Skewness need not mean disorder. It may indicate relationships that require a richer representation than the one initially chosen.**

## 6. Symmetry growth and contraction are different from symmetry breaking

A practical way to develop this is to examine a sequence of operations on a single seed polyhedron.



Cube Cuboctahedron Rhombic dodecahedron

Figure 5: Three structures with different incidence counts but the same full octahedral symmetry group.

Take the cube and apply familiar polyhedral operations:

| **Operation** | Result | What changes? |
|---|---|---|
| **Rectification** | Cuboctahedron | Vertices move to original edge midpoints; two face types appear |
| **Dualization** | Rhombic dodecahedron, dual of cuboctahedron | Vertices and faces exchange their combinatorial roles |
| **Truncation of cube** | Truncated cube | Original vertices become triangular faces; original faces become octagons |
| **Petrie transformation** | Petrial cube | Same vertices and edges, but the face circuits become skew hexagons |

Here are the first three structures compared:

| Structure | $V$ | $E$ | $F$ | $\lvert G\rvert$ |
|---|---|---|---|---|
| Cube | 8 | 12 | 6 | 48 |
| Cuboctahedron | 12 | 24 | 14 | 48 |
| Rhombic dodecahedron | 14 | 24 | 12 | 48 |

They share the same octahedral symmetry group, even though their incidence structures differ.

This leads to a major Formatic distinction:

$$\boxed{\text{Growth of form} \neq \text{growth of symmetry group}}$$

Indeed, symmetry might be preserved under an expansion of the number of vertices, edges, or faces. It may also be reduced while the combinatorial structure becomes richer.

We could therefore treat Formatic transformations as acting on multiple independent quantities:

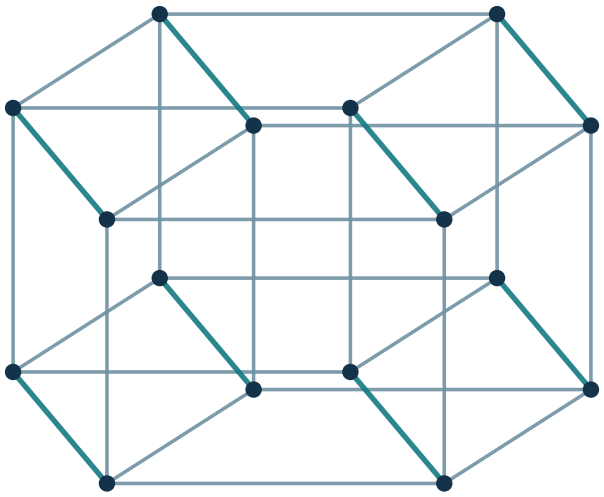
$$\mathcal{C}(\mathcal{P}) = (V, E, F, \chi, |G|, n_V, n_E, n_F, n_\Phi, \mathcal{H})\,.$$

Here $n_V, n_E, n_F$ count symmetry orbits of vertices, edges, and faces; $n_\Phi$ counts orbits of flags (incident vertex–edge–face triples); and $\mathcal{H}$ records selected holonomy data.

A regular polyhedron has one flag orbit. A noble polyhedron has one vertex orbit and one face orbit, but need not have one edge or flag orbit.

This is a more informative measure of symmetry complexity than counting faces or comparing group orders alone.

## 7. Higher-dimensional growth and finite groups



Schematic 2D projection of a 4D hypercube (tesseract)

Figure 6: Schematic planar projection of a four-dimensional hypercube. Projection complexity and intrinsic structure are distinct.

The transition to higher-dimensional polytopes offers another branching route.

For convex regular polytopes there are five in dimension three, six in dimension four, and just three families in each dimension five or higher.

But I would resist using dimensional growth as the primary definition of complexity.

A three-dimensional skew regular polyhedron may be more complicated in its incidence, symmetry, and transport structures than a conventional regular polytope realized in four dimensions.

For a group-theoretic starting point, the three finite Coxeter reflection groups associated with classical regular polyhedra are:

| Coxeter group | Order | Geometric context |
|---|---|---|
| $A_3$ | 24 | Tetrahedral symmetry |
| $B_3$ | 48 | Cubical/octahedral symmetry |
| $H_3$ | 120 | Dodecahedral/icosahedral symmetry |

Their rotation subgroups include $A_4$, $S_4$, and $A_5$, respectively. These offer a bridge between geometric polyhedral operations and finite-group representation theory.

For your Sporadic Group Framework, this suggests a concrete research program.

Rather than identifying an entire sporadic group with a geometric object at the outset, choose a subgroup $H$ acting on a finite informational complex.

Then study

$$H \curvearrowright \mathcal{P}$$

and ask how its geometry transforms through different subgroup actions, incidence constructions, quotient structures, and representations.

For example, the icosahedral rotational group $A_5$, with order 60, provides a natural seed. One can investigate how icosahedral symmetry appears inside larger finite groups, including appropriate subgroups or sections associated with sporadic groups.

This does not itself establish a physical identification. It supplies a mathematically testable route from group structure to polyhedral form.

The same strategy could be extended to finite groups of Lie type and to the sporadic groups, provided the incidence construction is specified for each case.

## 8. A proposed Formatic theory of symmetry routes

I would represent a Formatic object not merely by a polyhedron $P$, but by an enriched finite structure:

$$\boxed{\mathfrak{F} = (\mathcal{P}, G, \ell, \kappa, U, \iota)}$$

where:

- $\mathcal{P}$ is the finite incidence complex.

- $G$ is a specified symmetry action.
- $\ell$ is the internal metric data.
- $\kappa$ is a chosen discrete curvature assignment.
- $U$ is a group-valued connection for transport.
- $\iota$ is a geometric realization or embedding.

With that structure, a *route* becomes a transformation

$$\mathfrak{F}_0 \xrightarrow{T_1} \mathfrak{F}_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} \mathfrak{F}_n.$$

Each operation can preserve some information and change other information. Truncation may preserve the symmetry group while changing incidence. Duality exchanges ranks in the face structure. Petrie transformations alter face circuits, and conformal or curvature flows alter metric data.

A useful preliminary set of route operators would be:

| **Route** | Typical operations | Formatic role |
|---|---|---|
| **Expansion** | Truncation, subdivision, stellation | Growth of incidence and geometric complexity |
| **Contraction** | Identification, quotienting, selective deletion | Reduction of degrees of freedom |
| **Dualization** | Vertex–face exchange | Redistribution of structural roles |
| **Skew transformation** | Petrials, nonplanar circuits, helical extension | Change in realization or incidence organization |
| **Normalization** | Discrete curvature flow, metric adjustment | Redistribution of local geometry |
| **Transport** | Group-valued connections along cycles | Dynamic information and holonomy |

These routes do not all define a partial order. Dualization is often involutive, for example, and curvature evolution may follow a continuous parameter. But together they provide a network of transformations rather than a static classification.

There is also a subtle distinction between **infinite geometric extension** and **infinite information**.

An apeirohedron has infinitely many geometric elements, but a periodic example may have a finite fundamental domain and a finite description of its generating rules. Under a proposed finite-information interpretation, the finite quotient or generating data can be studied separately from the infinite realization. It is important not to identify the finite quotient with the complete infinite object.

That seems particularly compatible with your emphasis on finite information and rational mathematical representations.

## 9. The conceptual synthesis

I would phrase the emerging thesis this way:

> *Geometric form is the manifestation of organized finite relationships. Symmetry describes the transformations that preserve those relationships. Growth and contraction modify the incidence structure; conformal operations modify its metric; curvature normalization redistributes geometric constraints; and transport processes introduce information that can depend on complete routes through the structure. Skewness emerges when these relationships cannot be simultaneously accommodated within a chosen geometric representation.*

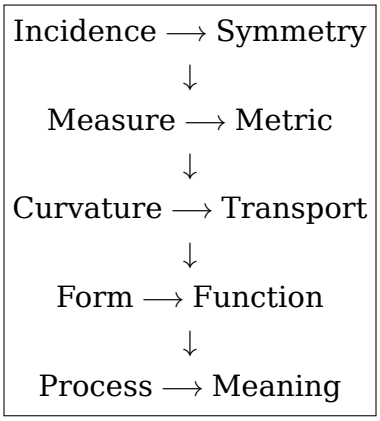
The most important conceptual move, to my mind, is to distinguish **four kinds of complexity**: incidence complexity, symmetry complexity, geometric realization complexity, and transport complexity.

They need not increase together.

This explains why the Johnson solids, Archimedean solids, Catalan solids, regular star polyhedra, 48 Grünbaum–Dress regular forms, and 146 noble exceptions are not naturally arranged along a single line from simple to complex. They occupy different branches of a multidimensional landscape.

For the next stage of your *Fundamentals of Measure and Meaning* project, I would begin with a **finite Formatic laboratory based on the cube, cuboctahedron, rhombic dodecahedron, and Petrial cube**. Those four provide a controlled set of symmetry-preserving transformations, changes in orbit structure, duality, and skew incidence. We could then introduce the finite curvature and holonomy constructions and move outward to the icosahedral, star-polyhedral, and sporadic-group regimes.

The longer-term conceptual sequence could become:




> **The provocative Formatic possibility is that symmetry is not simply a property of a completed form. Symmetry growth, contraction, skewness, and transport are processes through which form itself acquires relational information.**

That is a much richer foundation than merely counting regular solids.

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*Production note:* Figures in this PDF are original vector illustrations of the mathematical ideas, replacing the web-sourced visual examples displayed in the original chat. The analytical text and equations are retained.